Idealised GFD Models

Tim Leslie

Breakaway Consulting Pty. Ltd.
Climate Change Research Centre

May 11, 2012

Overview

Idealised GFD models let us investigate planetary physics.

- What kinds of things can we calculate?
- How do we map physical processes to mathematical statements?
- What do equations of a GFD model look like?

Equations of Planetary Fluid Dynamics

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - f(\mathbf{u} \times \hat{\mathbf{z}}) = -\nabla p - g\hat{\mathbf{z}} + A_2 \nabla^2 \mathbf{u} \qquad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \qquad (2)$$

- 2nd order differential equations $\Longrightarrow$ difficult to solve.
- Make simplifications based on physical assumptions.

Key assumptions

- Vertical velocity is small (compared to horiz. vel.)
- Fixed beta plane: $f(y) = f_0 + \beta(y - y_0)$
- Momentum diffusion is small (compare to pressure gradients)

Zeroth order approximations, QG balance

Pressure balances rotation:

$$f_0 v = p_x \quad (3)$$

$$f_0 u = -p_y \quad (4)$$

Hydrostatic assumption holds:

$$-g = p_z \quad (5)$$

These equations maintain continuity, as

$$u_x + v_y = 0. \quad (6)$$

Relative Vorticity

$$\zeta \equiv v_x - u_y \quad (7)$$

$$= \frac{\nabla_H^2 p}{f_0} \quad (8)$$

- A measure of the local rotation of the fluid.
- Relative to the earth.

First order equations

We wish to solve for u^*, v^*, using zero order values.

$$
\begin{aligned}
u_t + uu_x + vu_y - f_0 v^* - \beta(y - y_0)v &= -p_x^* + A_2 \nabla^2 u \quad (9) \\
v_t + uv_x + vv_y + f_0 u^* + \beta(y - y_0)u &= -p_y^* + A_2 \nabla^2 v \quad (10) \\
\nabla \cdot \mathbf{u}^* &= 0 \quad (11)
\end{aligned}
$$

Cross differentiate and subtract:

$$
\zeta_t + \left[u\left(\zeta + \beta(y - y_0)\right)\right]_x + \left[v\left(\zeta + \beta(y - y_0)\right)\right]_y = f_0 w_z^* + A_2 \nabla^2 \zeta \quad (12)
$$

Include planetary rotation:

$$
\begin{aligned}
q^* &\equiv \zeta + \beta(y - y_0) \quad (13) \\
q_t^* + (uq^*)_x + (vq^*)_y &= f_0 w_z^* + A_2 \nabla^2 \zeta \quad (14)
\end{aligned}
$$

Layer integral

Assume pressure and horizontal velocity are layer-averaged quantities.

$$q_t^* + (uq^*)_x + (vq^*)_y = \frac{f_0}{H}(w_+^* - w_-^*) + A_2\nabla^2\zeta \qquad (15)$$

Vertical velocity is a function of entrainment and variable layer thickness.

$$w^* = h_t + (uh)_x + (vh)_y + e \qquad (16)$$

$$= \eta_t + (u\eta)_x + (v\eta)_y + e \qquad (17)$$

where

$$\eta \equiv h - H. \qquad (18)$$

Pertubation Interface Height

$$\eta \equiv h - H \tag{19}$$

$$= \frac{(p_- - p_+)}{g'} \tag{20}$$

Reduced gravity is a constant,

$$g = g\left(\frac{\rho_- - \rho_+}{\rho_0}\right) \tag{21}$$

- $\eta = 0$ at top of atmosphere.
- $\eta = 0$ at atmosphere/ocean interface.
- $\eta = D$ at atmosphere/topography, ocean/bathymetry interface.

Put it all together...

$$q_t^* + (uq^*)_x + (vq^*)_y = -\frac{f_0}{H}\delta_z(w^*) + A_2\nabla^2\zeta \tag{22}$$

$$= -\frac{f_0}{H}\delta_z(\eta_t + (u\eta)_x + (v\eta)_y + e) + A_2\nabla^2\zeta \tag{23}$$

$$\left(q^* + \frac{f_0}{H}\delta_z(\eta)\right)_t + \left(u\left(q^* + \frac{f_0}{H}\delta_z(\eta)\right)\right)_x + \left(v\left(q^* + \frac{f_0}{H}\delta_z(\eta)\right)\right)_y = -\frac{f_0}{H}\delta_z(e) + A_2\nabla^2\zeta \tag{24}$$

Final QG Layer Equations

$$q \equiv q^* + \frac{f_0}{H}\delta_z(\eta) \quad (25)$$

$$= \frac{\nabla_H^2 p}{f_0} + \beta(y - y_0) + \frac{f_0}{H}\delta_z(\eta) \quad (26)$$

$$q_t = -(uq)_x - (vq)_y - \frac{f_0}{H}\delta_z(e) + A_2\frac{\nabla_H^4 p}{f_0} \quad (27)$$

Sources of Entrainment

- Interior convection (thermal entrainment)
- Ekman pumping in mixed layers.
  - Surface windstress
  - Bottom drag
  - $w_{ek} = -\int_{ek} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\, dz$
  - Geostrophic velocity + stress induced velocity

Bottom Drag

- Prescribed Ekman layer thickness $\delta_{ek} = \sqrt{K/|f_0|}$.

Velocities

$$(u, v) = \left(-\frac{p_y}{f_0} - \frac{\tau^y}{\delta_{ek} f_0}, \frac{p_x}{f_0} + \frac{\tau^x}{\delta_{ek} f_0}\right) \tag{28}$$

Linear stress

$$(\tau^x, \tau^y) = \frac{K}{\delta_{ek}} (u, v) \tag{29}$$

Solve for u, v

$$(u, v) = \left(-\frac{p_y}{2f_0} - \frac{p_x}{2|f_0|}, \frac{p_x}{2f_0} - \frac{p_y}{2|f_0|}\right) \tag{30}$$

Bottom Drag (2)

Derivatives of velocity

$$
\begin{aligned}
\frac{du}{dx} &= -\frac{p_{xy}}{2f_0} - \frac{p_{xx}}{2|f_0|} && (31)\\
\frac{dv}{dy} &= \frac{p_{xy}}{2f_0} - \frac{p_{yy}}{2|f_0|} && (32)
\end{aligned}
$$

Integrate over ekman layer

$$
\begin{aligned}
w_{ek} &= -\int_{ek} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\, dz && (33)\\
&= \delta_{ek}\left(\frac{p_{xx}}{2|f_0|} + \frac{p_{yy}}{2|f_0|}\right) && (34)\\
&= \frac{\delta_{ek}}{2|f_0|}\nabla_H^2 p && (35)
\end{aligned}
$$

Ocean Ekman Pumping

For a prescribed windstress vector $\hat{\tau}$, the induced mixed layer velocity is

$$(u, v) = \left(-\frac{p_y}{f_0} - \frac{\tau^y}{Hf_0}, \frac{p_x}{f_0} + \frac{\tau^x}{Hf_0}\right) \quad (36)$$

$$\begin{aligned} w_{ek} &= -\int_{ek} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\, dz && (37) \\ &= \int_{ek} \frac{\tau^y_x}{Hf_0} - \frac{\tau^x_y}{Hf_0}\, dz && (38) \\ &= \frac{\tau^y_x - \tau^x_y}{f_0} && (39) \\ &= \frac{\nabla \times \hat{\tau}}{f_0} && (40) \end{aligned}$$

Computing Windstress

$$({}^{a}u, {}^{a}v) = \left(-{}^{a}p_y - \frac{{}^{a}\tau^y}{{}^{a}Hf_0}, {}^{a}p_x + \frac{{}^{a}\tau^x}{{}^{a}Hf_0}\right) \quad (41)$$

$$({}^{o}u, {}^{o}v) = \left(-{}^{o}p_y - \frac{{}^{o}\tau^y}{{}^{o}Hf_0}, {}^{o}p_x + \frac{{}^{o}\tau^x}{{}^{o}Hf_0}\right) \quad (42)$$

$$({}^{a}\tau^x, {}^{a}\tau^y) = C_D \left|{}^{a}\mathbf{u} - {}^{o}\mathbf{u}\right| ({}^{a}u - {}^{o}u, {}^{a}v - {}^{o}v) \quad (43)$$

$${}^{o}\tau = \frac{{}^{a}\rho}{{}^{o}\rho}{}^{a}\tau \quad (44)$$

- We need to solve this quadratic(ish) equation
- A few pages of algebra later...

Windstress Equations

$$
\begin{aligned}
{}^{a}\tau^{x} &= C_D M \left[\frac{({}^{a}v^{*} - {}^{o}v^{*}) + (a+b)M({}^{a}u^{*} - {}^{o}u^{*})}{1 + (a+b)^2 M^2} \right] && (45) \\
{}^{a}\tau^{y} &= C_D M \left[\frac{({}^{a}u^{*} - {}^{o}u^{*}) - (a+b)M({}^{a}v^{*} - {}^{o}v^{*})}{1 + (a+b)^2 M^2} \right] && (46)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \frac{C_D}{{}^{a}Hf_0} && (47) \\
b &= \frac{{}^{a}\rho}{{}^{o}\rho} \frac{C_D}{{}^{o}Hf_0} && (48) \\
M &= \frac{1}{\sqrt{2}|a+b|} \sqrt{-1 + \sqrt{1 + 4(a+b)^2 |{}^{a}\mathbf{u}^{*} - {}^{o}\mathbf{u}^{*}|^2}} && (49)
\end{aligned}
$$

and we use geostrophic velocities

$$
(u^{*}, v^{*}) = (-p_y, p_x) \qquad (50)
$$

Oceanic Thermal Balance

- Introduce a mixed layer at the surface.
- Assume no deep ocean heat flux.
- Need to balance heat flux due to ekman pumping.

$$e\Delta T_{12} = -\frac{1}{2}\Delta T_{m1} w_{ek} \qquad (51)$$

Ocean Mixed Layer Temperature Evolution

Advection (geostrophic + stress) + diffusion + external forcing.

$$T_t + \nabla \cdot (\mathbf{u} T) = K_2 \nabla_H^2 T - K_4 \nabla_H^4 T - \frac{F_0 - F_e}{\rho C_p H} \tag{52}$$

$$T_t + (uT)_x + (vT)_y - \frac{w_{ek} T}{H} = K_2 \nabla_H^2 T - K_4 \nabla_H^4 T - \frac{F_0 - F_e}{\rho C_p H} \tag{53}$$

$$\begin{aligned} & T_t + (uT)_x + (vT)_y \\ = \quad & K_2 \nabla_H^2 T - K_4 \nabla_H^4 T - \frac{F_0}{\rho C_p H} + \frac{w_{ek}}{H} \left(T - \frac{T_{1m}}{2} \right) \end{aligned} \tag{54}$$

and F_0 is yet to be found.

Ocean convection

- If $T_m < T_1$ we get convection
- Need to determine a convective velocity w_c
- Need to balance convective heat flux with entrainment
- Need to adjust heat flux in temperature evolution equation.

$$e_c \Delta T_{12} = -\Delta T_{m1} w_c \quad (55)$$

$$(T_t)_c = \rho C_p \Delta T_{m1} w_c \quad (56)$$

Next Lecture...

- Moving from equations to a well defined system
- Knobs, dials and switches: controlling a system
- Using a system to answer scientific questions